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Minimal Projections in Bivariate Function Spaces

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1. INTRODUCTION

Suppose X is a real Banach space and Y a linear subspace of X. Then a projection from X onto Y is a linear map whose range is Y and which is idempotent on Y, i.e., $Py = y$ for all y in Y. In all cases it is possible to find a lower bound for the norm of any projection from a given Banach space X onto a fixed linear subspace Y. The greatest lower bound of this set is known as the projection constant of the given subspace with respect to the space in which it lies. Sometimes the word relative is used to indicate the fact that X is fixed. A projection whose norm is equal to this constant is called a minimal projection. In this generality there is no way of knowing whether such a projection exists or if it does, how it is characterised. However, if Y is finite-dimensional, then the existence question may be settled affirmatively.

In this paper we investigate some problems first discussed by Jameson and Pinkus [1]. They exhibited a minimal projection from $C(S \times T)$ onto $C(S) + C(T)$, where S and T are compact Hausdorff spaces each containing infinitely many points. In order to be sure that their projection was minimal they calculated the projection constant for the subspace $C(S) + C(T)$ and found it to be 3. In this paper we shall calculate the projection constant for the subspace $L_1(S) + L_1(T)$ as a subspace of $L_1(S \times T)$ and $L_\infty(S) + L_\infty(T)$ as a subspace of $L_\infty(S \times T)$. Note that some restrictions on the measure spaces S and T are inevitable. For example, in order that $L_1(S)$ and $L_1(T)$ can be regarded as subspaces of $L_1(S \times T)$ we need S and T to have finite measure. A further restriction, corresponding to the assumption that S and T contain infinitely many points in [1], will also be needed.

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II. The Finite-Dimensional Case

In this section we shall use $\mathbb{R}^n$ to denote the linear space of n-vectors. Points in $\mathbb{R}^{nm}$ can then be identified with matrices of size $n \times m$. We shall use both l_1 and l_∞-norms. In l_1^{nm} this can be given in its most general form by choosing a "weight matrix" $W = (w_{ij})$ such that $w_{ij} > 0$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, and $\sum_{i=1}^n \sum_{j=1}^m w_{ij} = 1$. Then for any matrix $A \in l_1^{nm}$ we define

$$\|A\|_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} |a_{ij}|.$$

In the same manner we define for $A \in l_\infty^{nm}$

$$\|A\|_\infty = \max_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} \frac{1}{w_{ij}} |a_{ij}|.$$

We shall consider the subspace $M \subseteq \mathbb{R}^{nm}$, where M "=" $\mathbb{R}^n + \mathbb{R}^m$. Here the obvious abuse of notation has occurred in that $\mathbb{R}^n + \mathbb{R}^m$ stands for the set of matrices which are the sum of a matrix whose columns are constant and a matrix whose rows are constant. It will be important at a later point to observe that if X is such a matrix then

$$x_{rs} = x_{ks} + x_{rl} - x_{kl}, \qquad 1 \leqslant r, k \leqslant n; 1 \leqslant s, l \leqslant m.$$

We shall consider projections from $\mathbb{R}^{nm}$ onto M and obtain a complete description of the minimal projection in certain cases. It will be important to observe that our normalization is such that l_1^{nm} and l_∞^{nm} are in duality. Now let P be a projection from $\mathbb{R}^{nm}$ onto M. It is convenient to describe the action of P (following [1]) on the matrices E_{rs}, which have zero entries everywhere except the (r, s) position where they are unity. We shall write

$$PE_{rs} = A_{rs} = (a_{ij}^{rs}).$$

Lemma 2.1.

$$\text{(i)} \quad \|P\|_\infty = \max_{i,j} \frac{1}{w_{ij}} \sum_{r=1}^{n} \sum_{s=1}^{m} |a_{ij}^{rs}| \, w_{rs},$$

$$\text{(ii)} \quad \|P\|_1 = \max_{r,s} \frac{1}{w_{rs}} \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}^{rs}| \, w_{ij}.$$

Proof. Part (i) is a straightforward computation, and (ii) may be obtained by recalling

$$\begin{aligned}\|P\|_1 = \|P^*\|_{(l_1^{nm})^*} &= \sup_{\substack{\|\phi\|_\infty = 1\\ \phi \in (l_1^{nm})^*}} \|P^*\phi\|_\infty \\ &= \sup_{\|\phi\|_\infty = 1} \max_{r,s} \frac{1}{w_{rs}} \left| \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^{rs} \phi_{ij} \right| \\ &\leqslant \max_{r,s} \frac{1}{w_{rs}} \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}^{rs}| \, w_{ij}.\end{aligned}$$

Equality is easily attainable at this final step.

In fact Lemma 2.1 is nothing more than the column and row sum formulae for l_1 and l_∞-norms of matrices when regarded as operators $\mathbb{R}^{nm} \to \mathbb{R}^{nm}$. Also, the result is not limited to projections but holds for any linear operator from $\mathbb{R}^{nm}$ to itself. In order that P should be a projection we require that

$$\text{(i)} \quad \sum_{r=1}^{n} a_{ij}^{rs} = \delta_{sj}, \qquad 1 \leqslant i \leqslant n; \quad 1 \leqslant s, j \leqslant m$$

$$\text{(ii)} \quad \sum_{s=1}^{m} a_{ij}^{rs} = \delta_{ri}, \qquad 1 \leqslant i, r \leqslant n; \quad 1 \leqslant j \leqslant m$$

$$\text{(iii)} \quad a_{ij}^{rs} = a_{kj}^{rs} + a_{il}^{rs} - a_{kl}^{rs}, \qquad 1 \leqslant r, i, k \leqslant n; \quad 1 \leqslant s, j, l \leqslant m.$$

From these three conditions we obtain

$$\begin{aligned}\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^{ij} &= \sum_{i=1}^{n} \sum_{j=1}^{m} \{a_{1j}^{ij} + a_{i1}^{ij} - a_{11}^{ij}\} \\ &= \sum_{j=1}^{m} \delta_{jj} + \sum_{i=1}^{n} \delta_{ii} - \sum_{i=1}^{n} \delta_{1i};\end{aligned}$$

i.e.,

$$\text{(iv)} \quad \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^{ij} = m + n - 1.$$

These four conditions appeared in [1].

THEOREM 2.2. *Let P be a projection from $\mathbb{R}^{nm}$ onto M. Then $\|P\|_1$ and $\|P\|_\infty$ are at least* $(1/nm \max_{i,j} w_{ij})\,(3 - 2\sum_{r=1}^{n} \sum_{s=1}^{m} a_{rs}^{rs} w_{rs})$ *and*

$$\left(\frac{\min_{i,j} w_{ij}}{\max_{i,j} w_{ij}}\right)\left(3 - 2 \sum_{r=1}^{n} \sum_{s=1}^{m} a_{rs}^{rs} w_{rs}\right) \textit{respectively.}$$

Proof. We consider first

$$\sum_{r,i=1}^{n} \sum_{s,j=1}^{m} w_{ij} |a_{ij}^{rs}|$$

$$\geqslant \sum_{r=1}^{n} \sum_{s=1}^{m} \left(- \sum_{\substack{i=1 \\ i \neq r}}^{n} \sum_{\substack{j=1 \\ j \neq s}}^{m} a_{ij}^{rs} w_{ij} + \sum_{j=1}^{m} a_{rj}^{rs} w_{rj} + \sum_{i=1}^{n} a_{is}^{rs} w_{is} - a_{rs}^{rs} w_{rs} \right)$$

$$= - \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{r=1}^{n} \sum_{s=1}^{m} a_{ij}^{rs} w_{ij} + 2 \sum_{j=1}^{m} \sum_{r=1}^{n} w_{rj} \sum_{s=1}^{m} a_{rj}^{rs}$$

$$+ 2 \sum_{i=1}^{n} \sum_{s=1}^{m} w_{is} \sum_{r=1}^{n} a_{is}^{rs} - 2 \sum_{r=1}^{n} \sum_{s=1}^{m} a_{rs}^{rs} w_{rs}$$

$$= -1 + 2 \sum_{j=1}^{m} \sum_{r=1}^{n} w_{rj} + 2 \sum_{i=1}^{n} \sum_{s=1}^{m} w_{is} - 2 \sum_{r=1}^{n} \sum_{s=1}^{m} a_{rs}^{rs} w_{rs}$$

$$= 3 - 2 \sum_{r=1}^{n} \sum_{s=1}^{m} a_{rs}^{rs} w_{rs}.$$

Hence there exist a pair (i_0, j_0) and a pair (r_0, s_0) such that

$$\sum_{r=1}^{n} \sum_{s=1}^{m} |a_{i_0 j_0}^{rs}| \geqslant 3 - 2 \sum_{r=1}^{n} \sum_{s=1}^{m} a_{rs}^{rs} w_{rs}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}^{r_0 s_0}| \, w_{ij} \geqslant \frac{1}{nm} \left(3 - 2 \sum_{r=1}^{n} \sum_{s=1}^{m} a_{rs}^{rs} w_{rs} \right)$$

and so

$$\|P\|_\infty \geqslant \frac{\min_{i,j} w_{ij}}{\max_{i,j} w_{ij}} \left(3 - 2 \sum_{r=1}^{n} \sum_{s=1}^{m} a_{rs}^{rs} w_{rs} \right)$$

$$\|P\|_1 \geqslant \frac{1}{nm \max_{i,j} w_{ij}} \left(3 - 2 \sum_{r=1}^{n} \sum_{s=1}^{m} a_{rs}^{rs} w_{rs} \right).$$

If the w_{rs} are all equal, i.e., $w_{rs} = 1/nm$, then we obtain a simpler version of Theorem 2.2.

COROLLARY 2.3. *Let* $w_{rs} = 1/nm$. *Then both* $\|P\|_\infty$ *and* $\|P\|_1$ *are at least* $3 - 2(n + m - 1)/nm$.

Proof. The result follows directly from Theorem 2.2 and condition (iv).

THEOREM 2.4. (i) *The minimal projection from l_1^{nm} onto M has norm $3-(2/nm)(n+m-1)$, when $w_{rs}=1/nm$,*

(ii) (*see* [1]). *The minimal projection from l_∞^{nm} onto M has norm $3-(2/nm)(n+m-1)$, when $w_{rs}=1/nm$.*

Proof. The given number is already a lower bound from Corollary 2.3 and the following projection (again from [1]) has the appropriate norm:

$$\begin{aligned} a_{ij}^{rs} &= -\frac{1}{nm}, && r\neq i, \quad s\neq j \\ &= \frac{n-1}{nm}, && r=i, \quad s\neq j \\ &= \frac{m-1}{nm}, && r\neq i, \quad s=j \\ &= \frac{n+m-1}{nm}, && r=i, \quad s=j. \end{aligned}$$

THEOREM 2.5. *The minimal projection from l_2^{nm} onto M has norm one and is identical with the minimal projections for l_1^{nm}, l_∞^{nm} given in the proof of the previous theorem.*

Proof. All that needs to be established here is that the appropriate projection is indeed the orthogonal projection. For this we write any point $A\in l_2^{nm}$ as $A=G+H+X$, where $G+H\in M$ and $X\in M^\perp$ if and only if $\sum_{i=1}^n x_{ij}=\sum_{j=1}^m x_{ij}=0$ for $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant m$. To see this suppose first X has the latter property. Then, if the rows of G are constant, $\langle G, X\rangle = \sum_i \sum_j g_{ij}x_{ij}=\sum_i g_{i1}\sum_j x_{ij}=0$. The proof for the columns of H being constant is similar. Alternatively, suppose $X\in M^\perp$. Then, for example, $\langle H, X\rangle=0$ for all matrices H which are constant along columns. This means that $\sum_i \sum_j h_{ij}x_{ij}=\sum_j h_{1j}\sum_i x_{ij}=0$, which gives one half of the required condition on X. The other half follows similarly. Now let P be the projection defined in the proof of Theorem 2.4. We shall indicate why $B-PB\in M^\perp$ for all $B\in l_2^{nm}$. For example, we must establish that $\sum_{i=1}^n (B-PB)_{ij}=0$ for $1\leqslant j\leqslant m$. Now

$$\begin{aligned} \sum_{i=1}^n (B-PB)_{ij} &= \sum_{i=1}^n \left(b_{ij}-\sum_{r,s} b_{rs}a_{ij}^{rs}\right) \\ &= \sum_{i=1}^n b_{ij}-\sum_{r,s} b_{rs}\sum_{i=1}^n a_{ij}^{rs}. \end{aligned}$$

Furthermore,

$$\sum_{i=1}^{n} a_{ij}^{rs} = \frac{-1}{nm}(n-1) + \frac{n-1}{nm}, \qquad s \neq j$$

$$= \frac{(m-1)(n-1)}{nm} + \frac{n+m-1}{nm}, \qquad s = j.$$

Thus $\sum_{i=1}^{n} (B-PB)_{ij} = \sum_{i=1}^{n} b_{ij} - \sum_{r=1}^{n} b_{rj} = 0.$

III. The Infinite-Dimensional Case

In this section we suppose that (S, Σ, μ) and (T, Θ, υ) are σ-finite measure spaces with $(S \times T, \Phi, \sigma)$ being constructed in the usual way. Via some lemmas and observations we aim to establish the following theorem.

THEOREM 3.1. (i) *Let S and T be σ-finite, non-atomic measure spaces. Then the minimal projections from $L_\infty(S \times T)$ onto $L_\infty(S) + L_\infty(T)$ have norm* 3 *and a minimal projection is given by*

$$(P_\infty f)(s, t) = \frac{1}{\mu_0} \int_{S_0} f(x, t)\, d\mu(x) + \frac{1}{\upsilon_0} \int_{T_0} f(s, y)\, d\upsilon(y) - \frac{1}{\mu_0 \nu_0} \iint_{S_0 \times T_0} f(x, y)\, d\sigma(x, y),$$

where S_0 and T_0 are any sets of finite measure in S and T, respectively, having measure μ_0 and υ_0.

(ii) *Let S and T be finite non-atomic measure spaces. Then the minimal projections from $L_1(S \times T)$ onto $L_1(S) + L_1(T)$ have norm* 3 *and a minimal projection is given by the same definition as in* (i), *where we may take $S_0 = S$ and $T_0 = T$. We denote this projection by P_1.*

We do not claim any originality for Theorem 3.1(i) since it can easily be obtained from results in [1], combined with the arguments given below.

The assumption that the measure spaces S and T are non-atomic allows us to take $S_1, S_2, \ldots, S_n$ in S and $T_1, T_2, \ldots, T_n$ in T, where n is any natural number and $\{S_i\}_1^n$, $\{T_i\}_1^n$ are pairwise disjoint measurable sets. We shall assume, by scaling S and T if necessary, that $\mu(S_i) = \upsilon(T_i) = 1/n$ for $1 \leqslant i \leqslant n$. Of course, non-atomicity is a convenient but not a necessary condition for the existence of such sets. Throughout this section it will be convenient to reserve the notation $l_1^{n^2}$ exclusively for $\mathbb{R}^{n^2}$ with the norm as defined in Section II having weights $w_{ij} = n^{-2}$. Similarly, $l_\infty^{n^2}$ will have norm with these same weights. For simplicity, we shall assume that $L_1(S)$, $L_1(T)$ and $L_1(S\times T)$ are understood to imply finite measure spaces, while $L_\infty(S)$, $L_\infty(T)$ and $L_\infty(S\times T)$ imply σ-finite measure spaces. We shall define maps $Q_1\colon L_1(S\times T)\to l_1^{n^2}$ and $R_1\colon l_1^{n^2}\to L_1(S\times T)$ by

$$(Q_1 f)_{i,j} = n^2 \iint_{S_i\times T_j} f(s,t)\,d\sigma, \qquad f\in L_1(S\times T),$$

$$(R_1 A)(s,t) = \sum_{i,j=1}^{n} a_{ij}\chi_{S_i\times T_j}, \qquad A\in l_1^{n^2}.$$

We can also use the same definitions for maps $Q_\infty\colon L_\infty(S\times T)\to l_\infty^{n^2}$ and $R_\infty\colon l_\infty^{n^2}\to L_\infty(S\times T)$.

LEMMA 3.2. (i) $\|Q_1\| = \|R_1\| = 1$.
(ii) $\|Q_\infty\| = n^2$ *and* $\|R_\infty\| = n^{-2}$.

Proof. We shall only establish (ii) since the computations are all straightforward. Firstly,

$$n^2 \sup_{\|f\|_\infty=1} \max_{i,j} |(Q_\infty f)_{ij}| \leqslant n^2 \sup_{\|f\|_\infty=1} \max_{i,j} n^2 \iint_{S_i\times T_j} |f(s,t)|\,d\sigma \leqslant n^2.$$

The function f given by $f(s,t)=1$ almost everywhere provides attainment. Secondly,

$$\begin{aligned}\sup_{\|A\|_\infty=1}\ \operatorname*{ess\,sup}_{(s,t)\in S\times T} |(R_\infty A)(s,t)| &= \sup_{\|A\|_\infty=1}\ \operatorname*{ess\,sup}_{(s,t)\in S\times T} \left|\sum_{i,j=1}^{n} a_{ij}\chi_{S_i\times T_j}\right| \\ &= \sup_{\|A\|_\infty=1}\ \max_{i,j} |a_{ij}| = n^{-2}.\end{aligned}$$

This completes the proof.

Now comes the result on which this section rests. It is purely algebraic in character.

LEMMA 3.3. (i) *Let P be a projection from $L_1(S\times T)$ onto $L_1(S)+L_1(T)$. Then Q_1PR_1 is a projection from $l_1^{n^2}$ onto M.*

(ii) *Let P be a projection from $L_\infty(S\times T)$ onto $L_\infty(S)+L_\infty(T)$. Then $Q_\infty PR_\infty$ is a projection from $l_\infty^{n^2}$ onto M.*

Proof. In either case there is no doubt that the range and domain of the given operator are $\mathbb{R}^{n^2}$. For convenience in the following few lines we shall use Q to denote Q_1 or Q_∞ and R to denote R_1 or R_∞. We begin by showing that the range of QPR lies in M. It will suffice to show

$$(QPRA)_{ij}+(QPRA)_{kl}=(QPRA)_{kj}+(QPRA)_{il}$$

where $1\leqslant i,\ j,\ k,\ l\leqslant n$ and $A\in\mathbb{R}^{n^2}$. Since P is a projection onto either $L_\infty(S)+L_\infty(T)$ or $L_1(S)+L_1(T)$ we know that if $PRA=y$ then y can be written as the sum of two univariate functions g (a function of s) and h (a function of t). Then

$$\begin{aligned}(QPRA)_{ij}+(QPRA)_{kl}&=n^2\iint_{S_i\times T_j}(g+h)\,d\sigma+n^2\iint_{S_k\times T_l}(g+h)\,d\sigma\\&=n\int_{S_i}g\,d\mu+n\int_{T_j}h\,dv+n\int_{S_k}g\,d\mu+n\int_{T_l}h\,dv\\&=n^2\iint_{S_i\times T_l}(g+h)\,d\sigma+n^2\iint_{S_k\times T_j}(g+h)\,d\sigma\\&=(QPRA)_{il}+(QPRA)_{kj}.\end{aligned}$$

To verify that QPR is a projection on M we shall content ourselves with showing that if A is a matrix whose rows are constant then $QPRA=A$. The full result then follows from a similar argument when A is constant along columns plus the usual linearity. So suppose $A=(a_{ij})$, $a_{ij}=k_i$, $1\leqslant j\leqslant n$. Then it is clear that $RA\in L_\infty(S)$ and so $PRA=RA$. Then it is immediate from the form of Q that $QRA=A$.

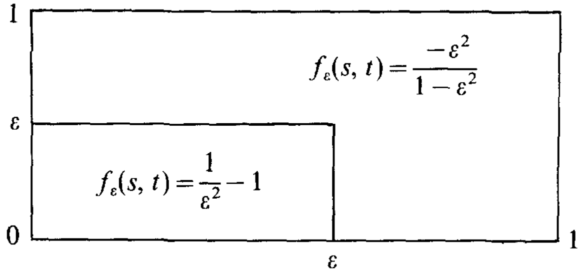
Proof of Theorem 3.1. Given any projection P from either $L_\infty(S\times T)$ onto $L_\infty(S)+L_\infty(T)$ or $L_1(S\times T)$ onto $L_1(S)+L_1(T)$ we may associate P with (respectively) a projection from $l_\infty^{n^2}$ onto M or $l_1^{n^2}$ onto M using the operators Q_1, Q_∞, R_1 and R_∞. Again writing Q for either Q_1 or Q_∞ and R for either R_1 or R_∞ an application of Corollary 2.3 gives

$$3-\frac{2}{n^2}(2n-1)\leqslant\|QPR\|\leqslant\|Q\|\ \|P\|\ \|R\|=\|P\|.$$

Since this inequality holds for all values of n we obtain $\|P\|\geqslant 3$. It is now elementary to verify from the definitions of P_1 and P_∞ that both have norm at most 3. This concludes the proof.

Notice that when S and T have finite measure we may choose S_0 and T_0 in the definition of P_∞ to coincide with S and T. In this case we can construct functions which are "nearly extremal" for P_1 and P_∞. Diagrammatically this is done in each case, when $S=T=[0,1]$.

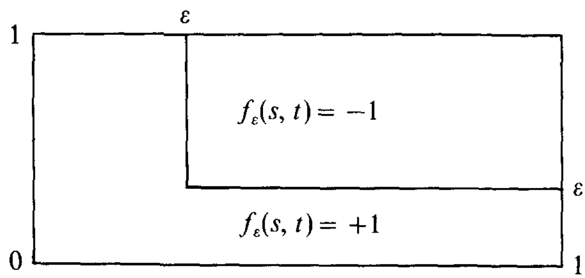
Case (i). $L_1(S \times T)$.



$$\|f_\varepsilon\|_1 = 1$$

$$\lim_{\varepsilon \to 0} \|P_1 f_\varepsilon\|_1 = 3.$$

Case (ii). $L_\infty(S \times T)$



$$\|f_\varepsilon\|_\infty = 1$$

$$\lim_{\varepsilon \to 0} \|P_\infty f_\varepsilon\|_\infty = 3.$$

IV. Remarks

A natural question to ask at this point is whether we can determine the projection constants for $L_p(S) + L_p(T)$ as subspaces of $L_p(S \times T)$, where $1 \leqslant p \leqslant \infty$. We have already dealt with the cases $p = 1, \infty$. The case $p = 2$ is, of course, the familiar Hilbert space case and the projection constant is necessarily unity there, with the usual orthogonal projection being the minimal projection. The dependence of the projection constant on p is an interesting question which is currently receiving attention.

We conclude with a brief comment about the difference between the problem in continuous function spaces and integrable function spaces. In [1] the transfer of the finite-dimensional results to the continuous function space was a matter of a simple identification. Once we lose the boundedness of the point evaluation functionals that identification becomes a little more intricate, as seen in Section III.

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Reference

1. G. J. O. Jameson and A. Pinkus, Positive and minimal projections in function spaces, *J. Approx. Theory* **37** (1983), 182–195.